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AN ELECTRONIC DIFFERENTIAL ANALYSER

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ABSTRACT. An electronic differential analyser having sixteen operational amplifiers is described. The analyser is specially designed for solving equations arising in connection with research works on circuit theory, servo-mechanism and electron trajectories. The analyser works both repetitively and non-repetitively. Different sources of error of the computer elements are also discussed. Accuracy of the solutions obtainable by the machine is illustrated by some examples.

INTRODUCTION

In the course of last ten years, electronic differential analysers have come to be recognised as useful tools for solving differential equations with moderate accuracy. (Ragazinni *et al.*, 1947; Williams and Ritson, 1947, Mc Nee, 1949; Meneley and Morill, 1953 ; Paul and Thomas, 1957, Biswas *et al.*, 1955). In the present paper is described an electronic differential analyser, built with a view to solving equations arising in connection with research work in circuit theory, servo mechanisms and electron trajectories. Very often interest is mainly centered on the qualitative rather than on the quantitative aspect of the solution. Hence in designing the computer emphasis was laid on low cost, moderate accuracy, ease of construction and flexibility.

The analyser can be used to solve both linear and non-linear equations. In the present paper only the linear computing elements are described. In the first part of the paper, the different sources of error in the computing elements and the factors governing the choice of the time scale of the computer are discussed. The actual hardware of the machine is described in part two. The accuracy obtainable by the machine is illustrated by examples. The non-linear components will be described in a paper to be published later

PART I

BASIC COMPUTING ELEMENTS FOR SOLVING LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT CO-EFFICIENTS

The basic computing elements required for solving linear differential equations with constant co-efficients are (1) adders, (2) integrators and (3) co-efficient setting potentiometers. Adders and integrators are constructed with operational amplifiers by the application of feedback. The general arrangement of feed back in the operational amplifier is shown in figure 1. $Z_1, Z_2, \dots Z_n$ are the input impedance elements and Z_f is the feedback impedance element. Output voltage

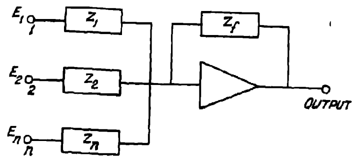


Fig. 1—Operational amplifier with feedback.

E_{out} of the amplifier due to the input voltages $E_1, E_2, \dots, E_n$ applied respectively at the n input terminals is

$$_{out} = -\left[\frac{Z_f}{Z_1}E_1 + \frac{Z_f}{Z_2}E_2 + \dots + \frac{Z_f}{Z_n}E_n\right] \quad \dots (1)$$

Operational amplifier is assumed to have ideal characteristics *i.e.* its gain, bandwidth and input impedance are infinitely large, output impedance infinitesimally small, and it has no drift.

When $Z_1 = Z_2 = \dots Z_n = Z_f$, E_{out} is the inverted sum of the voltages applied at the different inputs. The operational amplifier with this feedback arrangement is said to act as an adder. If, input is applied to only one terminal, it acts as an inverter.

When $Z_1 = Z_2 = \dots, Z_n = R$ (a resistance) and $Z_f = \frac{1}{pC}$ (*i.e.* the feedback impedance is due to a capacitance), E_{out} is the inverted sum of integrals of the inputs, scaled by CR. An operational amplifier with this feedback arrangement is referred to as an integrator.

A linear differential equation is solved by combining the adders and integrators to form a system whose response equation is the same as the equation to be solved. The coefficients of the equation are set by potentiometers.

Adder and Integrator errors.

E_{out} of an adder or integrator deviate from the ideal value given by Eq(1). due to,

(1) non-ideal behaviour of the operational amplifier.

(2) inaccuracy of the feedback and input network elements.

A practical amplifier has always a finite value of gain, bandwidth, input impedance or output impedance and also a finite drift. The exact equivalent circuit for the operational amplifier together with the network elements is shown in figure 2 (Mc. Donald, 1950)

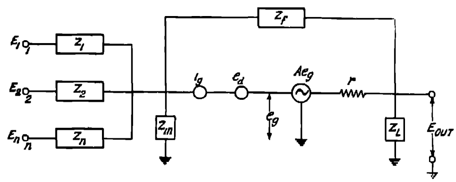


Fig. 2—Equivalent circuit of the operational amplifier with feedback.

Now,

$$E_{out} = -\frac{E_1\frac{Z_f}{Z_1} + E_2\frac{Z_f}{Z_2} + \dots + E_n\frac{Z_f}{Z_n} + e_d\left[1 + \frac{Z_f}{Z_1} + \dots + \frac{Z_f}{Z_{in}}\right] + Z_f i_g}{1 - \frac{1}{A'}\left[1 + \frac{Z_f}{Z_1}\dots + \frac{Z_f}{Z_n} + \frac{Z_f}{Z_{in}}\right]}$$

where $$A' = A \cdot \frac{Z'}{r + Z'},\; Z' = \frac{Z_L \times Z''}{Z_L + Z''},\; Z'' = Z_f + \frac{1}{\sum\limits_{i=1} \frac{1}{Z_i}}$$

A = the gain of the amplifier,
Z_{in} = the amplifier input impedance.
r = the amplifier output impedance.
Z_L = load impedance terminating the amplifier
i_g is a constant current generator to represent the drift due to grid current.
e_d is a constant voltage generator, with an output voltage e_d to represent the drift due to supply voltage variations, filament supply variations, contact p.d., amplifier parameter variations etc

If $$Z_1 = Z_2 \dots = Z_n = Z_0$$

$$E_{out} = -\frac{E_1 + E_2 \dots + E_n + Z_0 i_g + e_d\left[n + \frac{Z_0}{Z_f} + \frac{Z_0}{Z_{in}}\right]}{\frac{Z_0}{Z_f} - \frac{1}{A_1}} \tag{2}$$

where,

$$A_1 = \frac{A'}{n + \frac{Z_0}{Z_f} + \frac{Z_0}{Z_{in}}} \tag{3}$$

and

$$A' = A \frac{Z'}{r+Z'} , \; Z' = \frac{Z \times Z''}{Z_L+Z''} , \; Z'' = Z_f + \frac{Z_0}{n}.$$

Thus, the effect of the non-ideal behaviour of the operational amplifier is to introduce the term $\frac{1}{A_1}$ in the denominator, and $Z_0 \, i_g + e_d \left(n + \frac{Z_0}{Z_{in}} + \frac{Z_0}{Z_f}\right)$ in the numerator. Error due to the former may be called the response error while that due to latter is called the drift error.

Response error.

1. Frequency response error ·

Let $-|A_1(\omega)| e^{-j\phi(\omega)}$ represent the value of A_1 at any frequency ω. Substituting this in Eqn. 2

$$E_{out} = -\frac{E_1 + E_2 + \ldots + E_n + e}{\left[\left|\frac{Z_0}{Z_f}\right| \cos \psi + \left|\frac{1}{A_1(\omega)}\right| \cos\phi(\omega)\right] + j\left[\left|\frac{Z_0}{Z_f}\right| \sin \psi + \left|\frac{1}{A_1(\omega)}\right| \sin \phi(\omega)\right]} \quad \ldots(4)$$

where

$$\left|\frac{Z_0}{Z_f}\right| e^{j\psi} = \frac{Z_0}{Z_f}$$

and

$$e = Z_0 \, i_g + e_d\left[n + \frac{Z_0}{Z_f} + \frac{Z_0}{Z_{in}}\right].$$

Thus, E_{out} differs from the ideal in both amplitude and phase. The deviation in the amplitude normalised by the ideal value is called the amplitude error and the deviation in phase is the phase error.

For an adder $\left|\frac{Z_0}{Z_f}\right| = 1$ and $\psi = 0$, Hence, the output voltage of the adder is

$$E_{a\,(out)} = -\frac{E_1+E_2+\ldots+E_n+e}{\left|\frac{1}{A_1(\omega)}\right| \cos \phi(\omega)+1+j\left|\frac{1}{A_1(\omega)}\right| \sin \phi(\omega)}$$

The adder amplitude error, at any frequency ω is

$$\epsilon_a \, (\omega) = 1-\frac{1}{\sqrt{1+\frac{1}{|A_1(\omega)|^2}+\frac{2}{|A_1(\omega)|} \cos \phi(\omega)}} \approx -\frac{\cos \phi(\omega)}{|A_1(\omega)|} \quad \ldots \quad (5a)$$

Adder phase error $\delta\phi_a\ (\omega)$ is

$$\delta\phi\ (w) = \tan^{-1}\frac{\sin\phi(\omega)}{|A_1(\omega)| + \cos\phi(\omega)} \approx \tan^{-1}\frac{\sin\phi(\omega)}{|A_1(\omega)|}$$

$$\approx \frac{\sin\phi(\omega)}{A_1(\omega)} \qquad \ldots \quad (5b)$$

For an integrator

$$\left|\frac{Z_0}{Z_f}\right| = \omega cR = \frac{w}{\alpha_0},\ \alpha_0 = \frac{1}{CR} \text{ and } \quad \psi = \pi/2$$

Hence output of the integrator is

$$E_{out} = -[E_1+E_2+\ldots+E_n+e]\frac{1}{\dfrac{\cos\phi(\omega)}{|A_1(\omega)|}+j\left(\dfrac{\omega}{\alpha_0}+\dfrac{\sin\phi(\omega)}{|A_1(\omega)|}\right)}$$

Thus the integrator amplitude error at any frequency ω is

$$\epsilon_i\ (\omega) = 1-\frac{1}{\sqrt{1+\dfrac{1}{|A_1(\omega)|}\dfrac{\alpha_0{}^2}{\omega^2}+\dfrac{2\sin\phi(\omega)}{|A_1(\omega)|}\dfrac{\alpha_0}{\omega}}}$$

$$\approx \frac{1}{|A_1(\omega)|}\frac{\alpha_0}{\omega}\left[\sin\phi(\omega)+\frac{\alpha_0}{2\omega}\frac{1}{|A_1(\omega)|}\cdot\right] \qquad \ldots \quad (6a)$$

and integrator phase error is

$$\delta\phi_i\ (\omega) = \frac{\pi}{2} - \tan^{-1}\frac{\dfrac{\omega}{\alpha_0}+\dfrac{\sin\phi(\omega)}{|A_1(\omega)|}}{\dfrac{\cos\phi(\omega)}{|A_1(\omega)|}} \approx \frac{\cos\phi(\omega)}{\sin\phi(\omega)+|A_1(\omega)|\dfrac{\omega}{\alpha_0}} \qquad \ldots \quad (6b)$$

At frequencies for which $\qquad \dfrac{\omega}{\alpha_0} << 1$

$$\epsilon_i(\omega) \approx \frac{1}{2}\frac{\alpha^2_0}{\omega^2}\frac{1}{|A_1(\omega)|},\ \delta\phi_i(\omega) = \frac{\alpha_0}{\omega|A_1(\omega)|}$$

while at frequencies for which

$$\frac{\omega}{\alpha_0} >> 1$$

$$\epsilon_i(\omega) \approx \frac{\sin\phi(\omega)\,\alpha_0}{A_1(\omega)|\,\omega},\ \delta\phi_i(\omega) \approx \frac{\cos\phi(\omega)}{|A_1(\omega)|\,\omega/\alpha_0}$$

Now Z_L, the impedance terminating the amplifier is either due to a potentiometer or the input resistance element of an adder or integrator. When Z_L is due to a potentiometer $Z' = Z_L = R_p$, R_p is the resistance of a potentiometer and is assumed to be small compared to Z_f and $\frac{Z_0}{n}$. Hence $A' = A \frac{R_p}{r+R_p}$. On the other hand, when Z_L is due to input resistance of an adder or integrator $Z_L = R$, R is the resistance connected between the input terminal and grid of operational amplifier. So $A' = A \frac{R}{r+R}$. In both the cases A is multiplied by a constant factor which is very near unity.

Z_{in} is the input impedance of the operational amplifier and can be written as $\frac{1}{Z_{in}} = \frac{1}{R_{in}} + pC_{in}$. R_{in} and C_{in} being the input resistance and capacitance of the operational amplifier.

Hence A_1 is related to A by the relations

$$A_1 = \frac{A}{n+1+\frac{R}{R_{in}}+pRC_{in}} \quad \text{(for adder)} \qquad \ldots \quad (7a)$$

$$= \frac{A}{n+\frac{R}{R_{in}}+pR(C+C_{in})} \quad \text{(for integrator)} \qquad \ldots \quad (7b)$$

If the operational amplifier is a low pass one, $|A(\omega)|$ decreases with increase of frequency. Hence the largest error in the adder occurs at the highest operating frequency. A measure of the frequency response error of the adder can be obtained by knowing the highest frequency ω at which the amplitude error is ϵ_a % and denoted by $\omega_a(\epsilon)$. The integrator error, however, increases both at high and low frequencies. If $|A(\omega)|$ does not decrease very rapidly, the largest error in the integrator occurs at the lowest operating frequency. A measure of the integrator frequency error can be obtained by knowing the lowest frequency at which the integrator amplitude error is ϵ_i % and denoted by $\omega_i(\epsilon)$.

2. *Time Response Error* :

Adder error. A unit step applied to one of the inputs of the adder should ideally give at its output a unit step. But due to imperfection of the operational amplifier the output is different from a unit step. Determination of the actual output requires an exact knowledge of the frequency response characteristics of the operational amplifier. For a multistage, high gain d.c. amplifier exact frequency response characteristic is difficult to determine. However, in order that the adder be stable it is necessary that $|A_1(\omega)|$ be less than 1 when $\phi(\omega)$ approaches 180°. This is ensured by so designing the operational amplifier that A_1 can be taken to have effectively two poles. One of the poles being due to the input networks, operational amplifier contributes a single pole (say at α_1).

Thus A_1 can be written as

$$A_1 = -\frac{A_{10}\alpha_a\alpha_1}{(p+\alpha_a)(p+\alpha_1)} \text{ where } A_{10} = \frac{A_0}{(n+1+R/R_{in})} \tag{8a}$$

A_0 is the d.c. gain of the operational amplifier.

$$\alpha_a = \frac{\dfrac{1}{RC_{in}}}{n+1+R/R_{in}} \tag{8b}$$

Then, output of the adder E_{out} due to a unit step is as given below :

Case I: When $\quad (A_{10}+1)\alpha_a\ \alpha_1 < \left(\dfrac{\alpha_a+\alpha_1}{2}\right)^2$

$$E_{out} = -\frac{A_{10}}{(\alpha_1'-\alpha_2')(A_{10}+1)}\left[(\alpha_1-\alpha_2')+\alpha_2'e^{-\alpha_1't}-\alpha_1'e^{-\alpha_2't}\right]$$

α_1', α_2' are the roots of the equation.

$$p^2+(\alpha_a+\alpha_1)p+(A_{10}+1)\alpha_a\alpha_1 = 0$$

Case II : When $\quad (A_{10}+1)\ \alpha_a\alpha_1 = \left(\dfrac{\alpha_a+\alpha_1}{2}\right)^2$

$$E_{out} = -\frac{A_{10}}{(A_{10}+1)}\left[1-e^{-(\alpha_1+\alpha_a)t}\left\{1+(\alpha_1+\alpha_a)t\right\}\right]$$

Case III : When $\quad (A_{10}+1)\alpha_a\alpha_1 > \left(\dfrac{\alpha_a+\alpha_1}{2}\right)^2$

$$E_{out} = -\frac{A_{10}}{A_{10}+1}\left[1-\frac{\sqrt{(A_{10}+1)\alpha_a\alpha_1}}{\sqrt{(A_{10}+1)\alpha_a\sigma_1-\left(\dfrac{\alpha_a+\alpha_1}{2}\right)^2}}\ e^{-\frac{\alpha_a+\alpha_1}{2}t}.\right.$$

$$\left.\sin\left\{\left(\sqrt{(A_{10}+1)\alpha_a\alpha_1-\frac{\alpha_a+\alpha_1}{2}}\right)t+\tan^{-1}\frac{\sqrt{(A_{10}+1)\alpha_a\alpha_1-\left(\dfrac{\alpha_a+\alpha_1}{2}\right)^2}}{\dfrac{\alpha_a+\alpha_1}{2}}\right\}\right]$$

In all the three cases for large values of t, $E_{out} = -\frac{A_{10}}{A_{10}+1}$. Hence adder time response error for large values of t is

$$\epsilon_a(\text{large}) = \frac{1}{A_{10}} = \frac{n+1+R/R_{in}}{A_0} \qquad \ldots \quad (9a)$$

For small values of t, however, the error is much larger and the machine solution for small values of time has to be neglected. A measure of this small time error can be obtained by determining the error time, defined as the time measuring from zero after which the error is less than twice the large time error. Adder error time for the three cases can be obtained from

Case I. $$t_a \simeq \frac{1}{\alpha_2'} \ln \frac{1}{\epsilon_a\,(\text{large})} \quad \text{If } \alpha_1' \gg \alpha_2' \qquad \ldots \quad (9b)$$

Case II : $$t_a \simeq \ln \frac{1+(\alpha_a+\alpha_1)t_a}{\epsilon_a\,(\text{large})} \qquad \ldots \quad (9c)$$

Case III : $$t_a \simeq \frac{1}{\alpha_a+\alpha_1} \ln \frac{1}{\epsilon_a\,(\text{large})} \qquad \ldots \quad (9d)$$

Integrator Error.

The integrator should give a voltage $E_{out} = t/CR = \alpha_0 t$ at its output when a unit step is applied to one of its inputs (other inputs being grounded).

Now, for an integrator $$A_1 = -\frac{A_{10}\alpha_i\alpha_1}{(p+\alpha_i)(p+\alpha_1)}$$

where $$A_{10} = \frac{A_0}{n+R/R_{in}} \qquad \ldots \quad (10a)$$

$$\alpha_i = \frac{n+R/R_{in}}{R(C+C_{in})} \qquad \ldots \quad (10b)$$

Hence on neglecting small order terms, since it can be assumed that

$$\left(A_{10}\frac{\alpha_i}{\alpha_0}+1\right)\alpha_i+\alpha_1 \gg 4\alpha_i\alpha_1$$

$$E_{out} = A_{10}\left[1-e^{-\frac{\alpha_0}{A_{10}}t} - \frac{\alpha_0}{A}\,\frac{e^{-\left(\frac{A_{10}}{\alpha_0}\alpha_i+1\right)\alpha_1 t}}{\left(\frac{A_{10}\alpha_i}{\alpha_0}+1\right)\alpha_1}\right]$$

Also, when
$$\frac{\alpha_0}{A_{10}}\, t \leqslant 1$$

$$E_{out} \simeq A_{10}\left[\frac{\alpha_0}{A_{10}}\, t - \left(\frac{\alpha_0}{A_{10}} \right)^2 t^2 - \frac{\alpha_0}{A_{10}} \frac{e^{-\left(\frac{A_{10}\alpha_i}{\alpha_0}+1\right)\alpha_1 t}}{\left(\frac{A_{10}\alpha_i}{\alpha_0}+1\right)\alpha_1} \right]$$

Thus the integrator time response error is

$$c_i(t) = \frac{\alpha_0 t - E_{out}}{\alpha_0 t}$$

$$= \frac{\alpha_0}{A_{10}}\, t + \frac{e^{-\left(\frac{A_{10}}{\alpha_0}\alpha_i+1\right)\alpha_1 t}}{t\left[\frac{A_{10}\,\alpha_i}{\alpha_0}+1\right]\alpha_1}$$

$$= \frac{\alpha_0}{A_{10}}\, t + \frac{e^{-(A_0+1)\alpha_1 t}}{(A_0+1)\alpha_1 t}$$

The integrator error is, therefore, large both for small values and for large values of time For large values of t, ϵ_i (large) $= \frac{\alpha_0 t}{A_{10}}$ (13)

If T is the operation-time of the machine *i.e.* the time for which the solution is observed then ϵ_i (large) $= \frac{\alpha_0 T}{A_{10}}$ ... (14)

Error time t_i of the integrator is obtained from

$$\frac{\alpha_0}{A_{10}}\, t_i + \frac{e^{-(A_0+1)\alpha_1 t_i}}{(A_0+1)\alpha_1 t_i} = \epsilon_i \text{ (large)} \qquad \ldots \quad (15)$$

Drift Error.

It is seen from equation (2), that the effect of drift of the amplifier is to add a voltage $e = Z_0 i_g + e_d \left[n + \frac{Z_0}{Z_f} + \frac{Z_0}{Z_{in}} \right]$ to one input of the adder or integrator. In an adder the total drift error when referred to the inputs is thus $R_{ig} + (n+1 + R/R_{in})e_d$, e_d is assumed to vary slowly with time. Drift error for the integrator referred to the input is $R_{ig} + (n + R/R_{in})e_d$. The actual error in the output depends on the nature of variation of i_g and e_d with time.

Component Error.

In deducing Eq.(2) it was assumed that $Z_1 = Z_2 = \ldots = Z_n = Z_0$. When these impedances are due to elements whose value is accurate to within $\alpha\%$, in adding an input voltage an error of $2\alpha\%$ may be committed in the adder. Also the integrator equation was deduced assuming Z_f to be due to a loss-less condenser. However, condensers which are used have a finite loss which can be represented by assuming a resistance R_e to be present in parallel with the capacitance C.

The integrator output is then

$$E_{out} = -[E_1+E_2+ \dots +E_n + e]\frac{1}{pCR - \frac{1}{A_1} + \frac{R}{R_e}}$$

Hence the frequency response equations modify to

$$\epsilon_i(w) = \frac{1}{2}\frac{\alpha_0^2}{w^2}\frac{1}{|A_1(w)|} + \frac{\alpha_0}{w}\frac{\sin\phi(w)}{|A_1(w)|} + \frac{1}{2}\frac{\alpha_0^2}{w^2}\left(\frac{R}{R_e}\right)^2 + \frac{\cos\phi(w)}{|A_1(w)|}\frac{R}{R_e}\frac{\alpha_0^2}{w^2}$$

and

$$\delta\phi_i(w) = \frac{\cos\phi(w) + |A_1(w)|\, R/R_e}{|A_1(w)|\frac{w}{\alpha_0} + \sin\phi(w)}$$

The time response error equations are

$$\epsilon_i(\text{large}) = \frac{\alpha_0(1 + A_{10}\,R/R_e)}{A_{10}}\,T.$$

and

$$\frac{\alpha_0(1+A_{10}\,R/R_e)}{A_{10}}\,t_i + \frac{e^{-(A_0+1)\alpha_1 t_i}}{(A_0+1)\alpha_1 t_i} = \epsilon\,(\text{large})$$

Effect of the condenser leakage is to increase the error in the integrator low frequency response and also increasing the value of ϵ_i (large), for the same operation time.

Characteristics of the operational amplifier

The measured frequency response characteristics of the operational amplifier are given in figure 3 The operational amplifier has a d.c. gain of 12000, and the

Fig. 3—Frequency response characteristic of the operational amplifier.

first pole is at 30 c/s. Input capacitance is less than 10pf. Output impedance is 600 ohm. In or that the output impedance should not contribute a fraction of the adding resistance by more than 0.1%, the adder resistors should be made greater than 600K Ω. A value of 1M was hence chosen. In the following Table I is given the values of the adder and integrator errors calculated from the above data :

TABLE I

Adder Errors	No. of inputs			Used as scaler with scaling factor		
	$n = 1$			2	5	10
The highest frequency at which amplitude error is 1) less than 0.1% $\omega_a(.1)$				100 c/s	40 c/s	25 c/s
2) less than 1% $\omega_a(1)$	1.6 kc/s	1 kc/s	0 8 kc/s	1 kc/s	0 6 kc/s	300 c/s
Large time error	015%	025%	03%	025%	.05%	.1%
Adder error time	less than 10 μ sec.					

Integrator error	No. of inputs		
	$n = 1$	$n = 2$	$n = 3$
The lowest frequency at which the amplitude error is 1) less than 0.1% $\omega_i(.1)$	.08 $\alpha_0$rad/sec	.15 $\alpha_0$rad/sec	.25 $\alpha_0$rad/sec
2) less than 1% $\omega_i(1)$	008 $\alpha_0$rad/sec	.016 $\alpha_0$rad/sec	.025 $\alpha_0$rad/sec
Large time error in % (Operation time T)	$.008\alpha_0 T$	$.016\alpha_0 T$	$0\ 25\alpha_0 T$

Choice of the Computer Time scale

The above table has been prepared with a view to choosing the proper value α_0. It is seen that if the adders are required to scale by a factor of 10, the highest oscillation frequency of the system should be kept lower than 25 c/s. On the other hand, the lowest oscillation frequency ω should be such that $\frac{\omega}{\alpha_0}$ is greater than .25. Thus, to handle equations having characteristic frequencies, such that the ratio of the highest to lowest characteristic frequency is less than 100, α_0 has to be chosen to be equal to 6 rad/sec in order to ensure that the amplitude errors at the highest and lowest oscillation frequencies are less than .1%. The required value of the integrator capacitor works out to $1/6\mu F$. It should be noted that, when preparing a problem, it should be so arranged, that the lowest oscillation frequency is greater than $.25\alpha_0$ and the highest less than 25 c/s.

PART II

THE ANALYSER

General Description.

There are 16 operational amplifiers, four in one chasis, arranged as shown in the figure 4. Eight operational amplifiers are arranged to work as adders and eight as integrators.

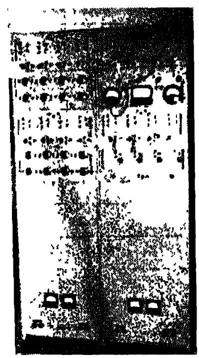


Fig. 4. A photograph of the electronic differential analyser.

There are provisions for three inputs and two outputs of the adders through jack. The input resistors and feedback resistors, matched to within 0.1% are wired in the chassis. But the feedback resistors can be disconnected from the input of the operational amplifier by disconnecting an external link between terminals numbered 2 and 3 in figure 5. Access to the input and the output of

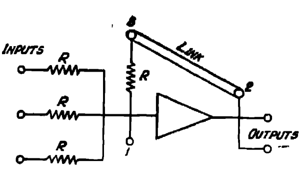


Fig. 5. Adder circuit.

the operational amplifier is provided by terminals 1 and 2 respectively. The arrangement enables one to replace the feedback resistor by another fixed resistor or potentiometer, or to connect input to the adder through any other resistor.

The integrator input resistors are also wired in the chassis. Each integrator has provisions for three inputs. Integrator capacitors are arranged in a different pannel. The integrator capacitors and resistors are matched to make the time constant of all the integrator equal to within .1%. Initial conditions are set in the machine by charging the integrator capacitors. Integrator circuit arrangement is shown in the figure 6. In the reset position the input of the operational amplifier is connected to terminal 2 and the integrator condenser is charged to the

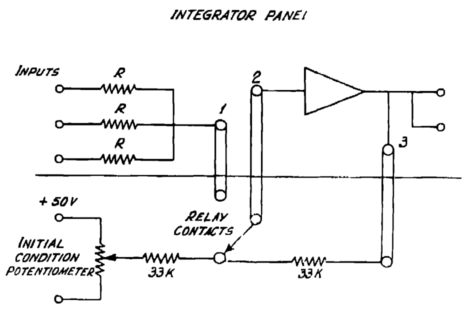


Fig. 6. Integrator circuit.

potential set by initial condition potentiometer In the compute position the input of the operational amplifier is connected to the common terminal of the input resistors. The terminals of the integrator are connected by external links to the initial condition panel terminals.

Switching of input between the terminals 1 and 2 is done by a relay, the arrangement of which is shown in figure 7. Relay 1 and relay 2 provide the eight one pole two-way contacts required for the 8 integrators. In the reset condition

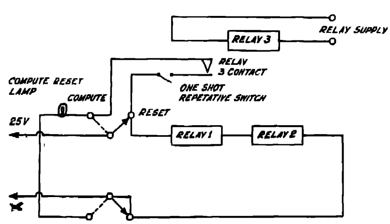


Fig. 7. Compute-reset relay circuit.

the relays are energised and while in the compute condition they are de-energised. Relay 3 makes the compute condition repetitive. The driving circuit for relay

3 is shown in figure 8. The repetition rate is determined by a multivibrator and can be set to 1 c/s, 0.5 c/s, 0.1 c/s. The computer is made repetitive or single shot by the one-shot-repetitive switch.

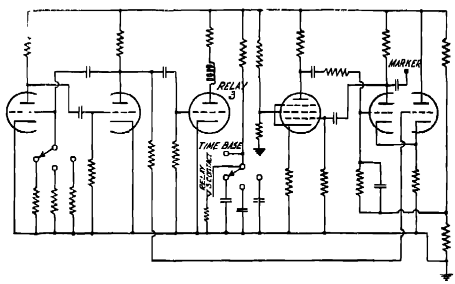


Fig. 8. Time base, marker and relay 3 driving circuit.

Coefficients are set by helipot potentiometers of 0.1% accuracy with loading error correction. The machine solution can be viewed in meters or in an oscilloscope having long persistance tube. For one-shot operation, the oscilloscope time base is worked on single shot while for repetitive operation the time base is supplied by the relay 3 driving circuit. The time is marked by time markers generated by a pulsed audio oscillator. Permanent record of the solution at present is made by photographing the oscilloscope output.

The values of the solution at different times are measured by an arrangement shown in figure 9. During the reset cycle inputs to the adder are disconnected by relays and the oscilloscope line indicates the zero level of its output. The value of the solution at any time is measured by adding to the solution a voltage which

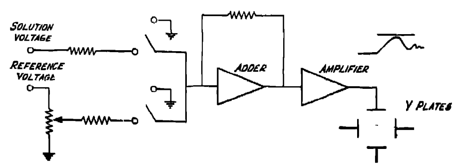


Fig. 9. A circuit for measuring the values of the solution at different times.

makes the oscilloscope spot coincide with the zero line at that time. With an amplifier gain of 100, the zero can be distinguished to within 10 mv. Also the voltage added being set by a helipot potentiometer can be measured to within 0.1%

Illustrative solutions.

In the following tables are given the solutions of three problems as obtained by the analyser.

In Table II is given percentage overshoot for different values of b in the equation,

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + y = H(t)$$

ep function.

TABLE II

b	% Overshoot	
	Calculated value	Experimentally obtained value
1	16 33	16 3
845	23.10	22 8
.684	31.86	31.7
.518	43.10	43 0
.347	57.50	57.4
.174	75.96	75 7
0	100.00	99.7

Table III gives the measured overshoot and rise time for Butterworth functions up to the sixth order. Figure 10 gives the response of the Butterworth functions due to a unit step.

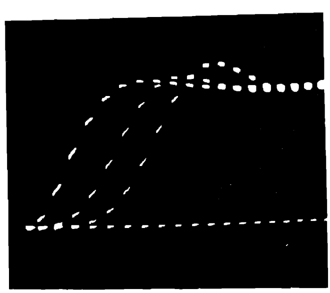


Fig. 10. Step response of Butterworth function of different orders.

TABLE III

Order of the function	% Overshoot	
	Calculated value	Experimentally obtained value
2	4 3	4.1
4	10.9	10.5
6	14 3	14.0

In Table IV is given the different combinations of the values of the coefficients in the equation

$$\frac{d^3y}{dt^3} + a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

which make the solution a continuous oscillatory function.

TABLE IV

a	b	c	
		Calculated value for maintained oscillation	Experimental value for maintained oscillation
3.00	2.00	6 00	6.06
3.00	3 00	9 00	8 99
3 00	4.00	12.00	12.02
3.00	5.00	15.00	14.99

It may be noted that the above equations were taken as representative of the types of equations that will be met with in practice. The results obtained indicate that whatever be the errors in the individual computing elements, the overall solution is accurate enough for practical purposes.

In order to illustrate the use of the differential analyser in solving electron trajectory problems, the motion of an electron when injected into a region containing a uniform electric field in the y-direction and a uniform magentic field in the Z-direction, is considered. The equations of motion of the electron can be written as

$$\ddot{y} = a - \omega\dot{x}$$

$$\ddot{x} = \omega\dot{y}$$

where

$$\omega = \frac{eH}{m} \quad \text{and} \quad a = \frac{eX}{m} \quad e, m, X \text{ and } H \text{ have their usual meaning.}$$

The set up of the computer for solving the above equations is shown in figure 11. Evidently, ω is equal to $1/CR$ and a is set by the magnitude of the input

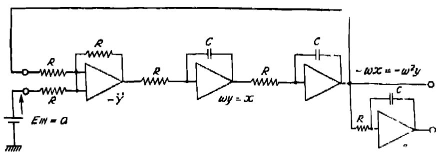
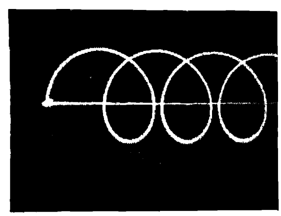


Fig. 11. Set up of the computer for obtaining the electron trajectory in crossed uniform electric and magnetic field.

voltage. Photographs of the trajectories for different initial velocities of the electron as obtained by the computer are given in figure 12.



(a) (b)

Fig. 12. Photographs of the electron trajectory as obtained by the computer.

$$\left.\begin{array}{l} \text{a) } \dot{x} = \dot{y} = 0 \\ \text{b) } \dot{x} = 0 \quad \dot{y} = \dot{y}_0 \end{array}\right\} \text{at } t = \sigma$$

CONCLUDING REMARKS

In solving a practical problem in circuit theory, servo-mechanism etc. with the differential analyser, the percentage overshoot, rise time and stability conditions are the important results that one is usually interested in. All differential analysers are such that the solutions obtained with them are accurate only over a certain range of values of time or frequency. But, by properly chosing the time scale of the computer, one can determine the overshoot, rise time or stability with sufficient accuracy.

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